

On the extension to slip boundary conditions of a Bae and Choe regularity criterion for the Navier-Stokes equations. The half-space case.

H. Beirão da Veiga *,
Department of Mathematics,
Pisa University, Italy.
email: bveiga@dma.unipi.it

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Abstract

This notes concern the sufficient condition for regularity of solutions to the evolution Navier-Stokes equations known in the literature as Prodi-Serrin's condition. H.-O. Bae and H.-J. Choe proved in a 1999 paper that, in the whole space \mathbb{R}^3 , it is merely sufficient that two components of the velocity satisfy the above condition. Below, we extend the result to the half-space case \mathbb{R}_+^n under slip boundary conditions. We show that it is sufficient that the velocity component *parallel* to the boundary enjoys the above condition. Flat boundary geometry seems not essential, as suggested by some preliminary calculations in cylindrical domains.

1 Introduction.

These notes concern sufficient conditions for regularity of solutions to the evolution Navier-Stokes equations related to the so called Prodi-Serrin's condition, see (3). In reference [1] the authors proved, in the whole space case, that it is sufficient that two components of the velocity satisfy the above condition. Below we extend this result to the half-space case \mathbb{R}_+^n under slip

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boundary conditions. The structure of the proof follows Bae and Choe paper, by adding a suitable control of some boundary integrals which, clearly, were not present in the whole space case.

We assume that readers are acquainted with the main literature on the subject. In particular, we will not repeat well know notation as, for instance, Sobolev spaces notation, and so on.

In the sequel we are interested in the evolution Navier-Stokes equations in the half-space $\mathbb{R}_+^n = \{x : x_n > 0\}$, $n \geq 3$,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \mu \Delta u + \nabla \pi = 0, \\ \nabla \cdot u = 0 \quad \text{in } \mathbb{R}_+^n \times (0, T]; \\ u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}_+^n, \end{cases} \quad (1)$$

under the classical Navier slip boundary conditions without friction. See [12] and [14]. On flat portions of the boundary this condition reads

$$\begin{cases} u_n = 0, \\ \partial_n u_j = 0, \quad 1 \leq j \leq n-1. \end{cases} \quad (2)$$

In the half-space case we will use this formulation. Let us recall that, for $n = 3$, the above slip boundary condition may also be written in the form $u_n = 0$, plus $\omega_j = 0$, for $j = 1, 2$, where $\omega = \nabla \times u$ is the vorticity field.

It is well know that weak solutions u satisfying the so called Prodi-Serrin's condition

$$u \in L^q(0, T; L^p(\mathbb{R}_+^n)), \quad \frac{2}{q} + \frac{n}{p} \leq 1, \quad p > n \quad (3)$$

are strong, namely

$$u \in L^\infty(0, T; H^1(\mathbb{R}_+^n)) \cap L^2(0, T; H^2(\mathbb{R}_+^n)). \quad (4)$$

The proof is classical. Furthermore, strong solutions are smooth, if data and domain are also smooth.

It is well known that the above results hold in a very large class of domains Ω , under suitable boundary conditions. We assume this kind of results well known to the reader. In particular, the result is well known in the whole space \mathbb{R}^n , which is our departure point. In fact, consider the Navier-Stokes equations (1) with \mathbb{R}_+^n replaced by \mathbb{R}^n . Differentiating both sides of the first equation (1) with respect to x_k , taking the scalar product with $\partial_k u$, adding over k , and integrating by parts over \mathbb{R}^n , one shows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 dx + \mu \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx = - \int_{\mathbb{R}^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u dx, \quad (5)$$

where obvious integrations by parts have been done. Clearly, no boundary integrals appear. A last integration by parts shows that

$$\left| \int_{\mathbb{R}^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u \, dx \right| \leq c(n) \int_{\mathbb{R}^n} |u| |\nabla u| |\nabla^2 u| \, dx. \quad (6)$$

So

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \mu \int_{\mathbb{R}^n} |\nabla^2 u|^2 \, dx \leq c(n) \int_{\mathbb{R}^n} |u| |\nabla u| |\nabla^2 u| \, dx \quad (7)$$

follows.

By appealing to the Prodi-Serrin's assumption (3) applied to the term $|u|$ present in the right hand side of (7), well known devices lead to the desired regularity result (4) in \mathbb{R}^n (these same devices may be seen in section 2, in connection with the similar estimate (21)). In the proof, the crucial property is that the term $|u|$ in the right hand side of estimate (6) (hence also in (7)) enjoys the Prodi-Serrin's condition. In reference [1], see also [2], H.-O. Bae and H.-J. Choe succeed in replacing, in the right hand side of (6), the term $|u|$ simply by $|\bar{u}|$, where \bar{u} is an arbitrary $n-1$ dimensional component

$$\bar{u} = (u_1, \dots, u_{n-1}, 0) \quad (8)$$

of the velocity u . In other words, they succeed in improving the quite obvious estimate (6), by showing the much stronger estimate

$$\left| \int_{\mathbb{R}^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u \, dx \right| \leq c(n) \int_{\mathbb{R}^n} |\bar{u}| |\nabla u| |\nabla^2 u| \, dx. \quad (9)$$

Hence the estimate (7) holds with $|u|$ replaced by $|\bar{u}|$. The classical $|u|$ -proof applies as well after this substitution. In this way the authors proved that (4) holds if merely \bar{u} (instead of u) satisfies the Prodi-Serrin's condition. A quite unexpected result, at that time, may be not yet sufficiently exploited. Clearly, in the whole space case, \bar{u} may be any $n-1$ dimensional component of the velocity.

The proof of the estimate (9) is based on a clever analysis of the structure of the integral on the left hand side of this equation.

The first aim of these notes is to prove equation (9) in the half space \mathbb{R}_+^n

$$\left| \int_{\mathbb{R}_+^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u \, dx \right| \leq c(n) \int_{\mathbb{R}_+^n} |\bar{u}| |\nabla u| |\nabla^2 u| \, dx, \quad (10)$$

under slip boundary conditions. As a consequence, the estimate (7) holds with $|u|$ replaced by $|\bar{u}|$ and \mathbb{R}^n replaced by \mathbb{R}_+^n . It readily follows, as in the classical case, that solutions to the above boundary value problem are regular provided that \bar{u} satisfies the Prodi-Serrin's condition (3).

Theorem 1.1. *Let u be a solution to the Navier-Stokes equations (1) in \mathbb{R}_+^n under the slip boundary conditions (2). Furthermore, let \bar{u} be the parallel to the boundary component of the velocity u , given by (8). If*

$$\bar{u} \in L^q(0, T; L^p(\mathbb{R}_+^n)), \quad \frac{2}{q} + \frac{n}{p} \leq 1, \quad p > n, \quad (11)$$

then (4) holds.

Alternatively, the proof of the above result could be done by appealing to a reflection principle, see [7]. However this does not help possible extensions to non flat boundaries. In this direction, in a forthcoming paper, we will consider the cylindrical coordinates case.

We end this section by quoting the very recent paper [4] where the authors proved the local, interior, regularizing effect of the Prodi-Serrin's condition only on two velocity components. It would be of interest to extend this result to arbitrary, smooth, coordinates (orthogonal for instance).

2 Extension to boundary value problems

In this section we prove equation (10). Our approach adds to that followed in reference [1] an accurate control of the boundary integrals, clearly not present in the whole space case. To obtain the explicit form of these integrals, we have to turn back to the volume integrals.

For notational convenience we set

$$\Gamma = \{x : x_n = 0\}.$$

The first step is to prove (5), now with \mathbb{R}^n replaced by \mathbb{R}_+^n , namely

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+^n} |\nabla u|^2 dx + \mu \int_{\mathbb{R}_+^n} |\nabla^2 u|^2 dx = - \int_{\mathbb{R}_+^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u dx. \quad (12)$$

By following the \mathbb{R}^n case, we differentiate both sides of the first equation (1) with respect to x_k , take the scalar product with $\partial_k u$, add over k , and

integrate by parts over \mathbb{R}_+^n . Now additional boundary integrals appear. We start from the Δu term. One has

$$- \int_{\mathbb{R}_+^n} \nabla(\Delta u) \cdot \nabla u \, dx \equiv - \int_{\mathbb{R}_+^n} \partial_k (\partial_i^2 u_j) \partial_k u_j \, dx = \int_{\mathbb{R}_+^n} |\nabla^2 u|^2 \, dx - I$$

where

$$I \equiv \int_{\Gamma} (\partial_i \partial_k u_j) (\partial_k u_j) \nu_i \, d\Gamma = - \int_{\Gamma} (\partial_k \partial_n u_j) (\partial_k u_j) \, d\Gamma,$$

since ν , the unit external normal to Γ , has components $(0, \dots, 0, -1)$. If $j < n$ and $k = n$ the terms $\partial_k u_j$ vanishes, due to the boundary conditions (2). If $j < n$, but $k < n$, the terms $\partial_k \partial_n u_j$ vanishes, since $\partial_n u_j = 0$ on the boundary, and ∂_k is a tangential derivative. Hence we merely have to consider the $j = n$ terms, namely $(\partial_n \partial_k u_n) \partial_k u_n$. If $k < n$, it follows $\partial_k u_n = 0$. On the other hand, if $k = n$, by appealing to the divergence free condition, one has

$$\partial_n \partial_n u_n = - \sum_{j < n} \partial_n (\partial_j u_j) = 0, \quad (13)$$

since $\partial_j (\partial_n u_j) = 0$ on Γ , for $j < n$. We have shown that

$$- \mu \int_{\mathbb{R}_+^n} \nabla(\Delta u) \cdot \nabla u \, dx = \mu \int_{\mathbb{R}_+^n} |\nabla^2 u|^2 \, dx.$$

the boundary integral related to the viscous term vanishes.

Next we consider the pressure term. One has, by an integration by parts,

$$\int_{\mathbb{R}_+^n} (\nabla(\nabla \pi)) \cdot \nabla u \, dx \equiv \int_{\mathbb{R}_+^n} \partial_k (\partial_j \pi) \partial_k u_j \, dx = - \int_{\mathbb{R}_+^n} (\nabla \pi) \cdot \nabla (\nabla \cdot u) \, dx + A,$$

where

$$A \equiv \int_{\Gamma} (\partial_k \pi) (\partial_k u_j) \nu_j \, d\Gamma = - \int_{\Gamma} (\partial_k \pi) \partial_k u_n \, d\Gamma = - \int_{\Gamma} (\partial_n \pi) (\partial_n u_n) \, d\Gamma,$$

since $\partial_k u_n = 0$ on the boundary for $k < n$. Furthermore, the volume integral on the right hand side vanishes, due to the divergence free condition.

Let's see that $A = 0$ by showing that $\partial_n \pi = 0$ on Γ . By appealing to the n -th equation (1) we show that $\partial_n \pi = -\partial_t u_n - (u \cdot \nabla) u_n + \mu \Delta u_n$. So, by appealing to boundary condition $u_n = 0$, one easily shows that

$$\partial_n \pi = \mu \Delta u_n \quad \text{on } \Gamma.$$

Note that $(u \cdot \nabla) u_n = 0$ on Γ . By taking into account that the second order tangential derivatives of u_n vanish on the boundary, we show that $\Delta u_n = 0$, by appealing to (13). So $\partial_n \pi = 0$ on Γ , as desired. We have shown that

$$\int_{\mathbb{R}_+^n} (\nabla(\nabla \pi)) \cdot \nabla u \, dx = 0. \quad (14)$$

Equation (12) is proved.

Note that equation (14) holds under the non-slip boundary condition, with a simpler proof. In fact, in this case, $A = 0$ follows immediately from $\partial_n u_n = 0$ on Γ , which is an immediate consequence of the divergence free property and the non-slip boundary assumption.

The next, and main, step is to consider the non-linear term. We start by showing that

$$\int_{\mathbb{R}_+^n} \nabla [(u \cdot \nabla) u] \cdot \nabla u \, dx = \int_{\mathbb{R}_+^n} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) \, dx. \quad (15)$$

This follows from the identity

$$\nabla [(u \cdot \nabla) u] \cdot \nabla u = (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) + \frac{1}{2} u_i \partial_i \left(\sum_{j,k} (\partial_k u_j)^2 \right) \quad (16)$$

since, by an integration by parts, we show that the integral of the second term on the right hand side of the (16) vanishes, as follows from the divergence free and the tangential to the boundary properties (unfortunately, in the cylindrical coordinates case, the counterpart of this main point is much more involved).

Next we prove the main estimate (10). Following [1], we consider separately the three cases $i \neq n$; $i = n$ and $j \neq n$; $i = j = n$.

If $i \neq n$, one has

$$\begin{aligned} & \int_{\mathbb{R}_+^n} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) \, dx = \\ & - \int_{\mathbb{R}_+^n} u_i \partial_k ((\partial_k u_j)(\partial_i u_j)) \, dx + \int_{\Gamma} u_i (\partial_k u_j)(\partial_i u_j) \nu_k \, dx \end{aligned} \quad (17)$$

The boundary integral is equal to

$$- \int_{\Gamma} u_i (\partial_n u_j)(\partial_i u_j) \, dx.$$

If $j \neq n$, one has $\partial_n u_j = 0$. If $j = n$, one has $\partial_i u_j = 0$, since ∂_i is a tangential derivative and $u_n = 0$. Hence the boundary integral in equation (17) vanishes. On the other hand, since $i \neq n$, the volume integral on the right hand side of equation (17) is bounded by the right hand side of inequality (10). After all, if $i \neq n$, the left hand side of equation (17) is bounded by the right hand side of inequality (10).

Next we assume that $i = n$ and $j \neq n$. In this case, by an integration by parts, one gets

$$\begin{aligned} & \int_{\mathbb{R}_+^n} (\partial_k u_i)(\partial_i u_j)(\partial_k u_j) dx = \\ & - \int_{\mathbb{R}_+^n} (\Delta u)(\partial_i u_j) u_j dx \\ & - \int_{\mathbb{R}_+^n} (\partial_k u_n)(\partial_i \partial_k u_j) u_j dx, \end{aligned} \quad (18)$$

since the boundary integral, which appears after the above integration by parts, vanishes. In fact, the terms $(\partial_i u_j)$ vanish on the boundary, for $i = n$ and $j \neq n$. From (18) it follows that the left hand side of this equation is bounded by the right hand side of inequality (10), as desired.

If $i = j = n$, we have to estimate the integral

$$B \equiv \int_{\mathbb{R}_+^n} (\partial_k u_n)^2 (\partial_n u_n) dx = - \int_{\mathbb{R}_+^n} (\partial_k u_n)^2 \left(\sum_{j \neq n} \partial_j u_j \right) dx.$$

By integration by parts one gets

$$B = 2 \int_{\mathbb{R}_+^n} (\partial_k u_n) \left(\sum_{j \neq n} \partial_j \partial_k u_n \right) u_j dx - \int_{\Gamma} (\partial_k u_n)^2 \sum_{j \neq n} u_j \nu_j d\Gamma.$$

Since the above boundary integral vanishes, the absolute value of B is bounded by the right hand side of inequality (10). The proof of (10) is accomplished. From now on the proof of theorem 1.1 follows a very classical way. For the readers' convenience we recall how to prove (5). From (12) and (10) it follows that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}_+^n} |\nabla u|^2 dx + \mu \int_{\mathbb{R}_+^n} |\nabla^2 u|^2 dx \leq c(n) \| |\bar{u}| \nabla u \|_2 \| \nabla^2 u \|_2. \quad (19)$$

On the other hand, by Hölder's inequality,

$$\| |\bar{u}| \nabla u \|_2 \leq \| \bar{u} \|_p \| \nabla u \|_{\frac{2p}{p-2}}.$$

Furthermore, by interpolation and Sobolev's embedding theorem,

$$\|\nabla u\|_{\frac{2p}{p-2}} \leq \|\nabla u\|_2^{1-\frac{n}{p}} \|\nabla u\|_{2^*}^{\frac{n}{p}} \leq c \|\nabla u\|_2^{1-\frac{n}{p}} \|\nabla^2 u\|_2^{\frac{n}{p}},$$

since $(p-2)/(2p) = (1-n/p)/2 + (n/p)/2^*$. Here $2^* = 2n/(n-2)$ is a well known Sobolev's embedding exponent (note that each single component of the tensor ∇u satisfies an homogeneous, Dirichlet or Neumann, boundary condition on Γ). Consequently,

$$\|\bar{u}\|_2 \|\nabla u\|_2 \|\nabla^2 u\|_2 \leq c \|\bar{u}\|_p \|\nabla u\|_2^{1-\frac{n}{p}} \|\nabla^2 u\|_2^{1+\frac{n}{p}}.$$

Hence, by Young's inequality,

$$\|\bar{u}\|_2 \|\nabla u\|_2 \|\nabla^2 u\|_2 \leq c \leq c \|\bar{u}\|_p^q \|\nabla u\|_2^2 + (\mu/2) \|\nabla^2 u\|_2^2. \quad (20)$$

From (21) and (20) we get, for $t \in (0, T]$,

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_2^2 + \frac{\mu}{2} \|\nabla^2 u\|_2^2 \leq c \|\bar{u}\|_p^q \|\nabla u\|_2^2. \quad (21)$$

This estimate immediately leads to (5) since, by the Prodi-Serrin's assumption,

$$\|\bar{u}\|_p^q \in L^1(0, T).$$

3 Non flat boundaries.

It would be of basic interest to understand how crucial is in the theorem 1.1 the flat-boundary hypothesis. Does the result hold in the neighborhood of non-flat boundary points? We propose to start from the following particular case.

Open problem 3.1. Let (ρ, θ, z) be the canonical cylindrical coordinates in the three dimensional space, and consider the subset defined by imposing to ρ the constraint $r < \rho < R$, where r and R are positive constants. Assume the slip boundary condition on the two lateral cylindrical surfaces, and space periodicity with respect to the axial z -direction. To *prove* or *disprove* that Prodi-Serrin's condition on the two "tangential" components u_θ and u_z implies smoothness.

A careful, preliminary, study of the above problem showed that the possible main obstacles still appear in a $2-D$ approach in plane-polar coordinates. Furthermore, calculations in this direction led us to be convinced that the reply to the open problem 3.1 is positive.

4 The limit case $p = n$.

In the following $\|\cdot\|_p$ denotes the L^p norm, moreover $\|\cdot\| = \|\cdot\|_2$. The Prodi-Serrin's condition for $(q, p) = (\infty, n)$, namely

$$u \in L^\infty(0, T; L^n(\Omega)), \quad (22)$$

deserves a separate treatment. In reference [15] uniqueness of solutions was proved under assumption (22). In references [9] and [16] strong regularity was proved by assuming time-continuity in $L^n(\Omega)$.

In reference [5] the system (1) was considered in a smooth bounded domain Ω , under the *non-slip* boundary condition. It was shown that, under assumption (22), the addition of a sufficiently small upper-bound for possible left-discontinuities on the norm $\|u(t)\|_n$ implies regularity (the same result was proved in the same year, in reference [11]). Note that, under assumption (22), weak solutions are weakly continuous with values in $L^n(\Omega)$. Hence the pointwise values $\|u(t)\|_n$ are everywhere well defined. In reference [5] a very simple explicit upper bound for the required left-discontinuities was also shown. It was proved that there is a positive, independent, constant C such that if (22) holds and, in addition,

$$\limsup_{t \rightarrow \bar{t}-0} \|u(t)\|_n < \|u(\bar{t})\|_n + 4(C/4)^n \quad (23)$$

for each $\bar{t} \in (0, T]$, then solutions are smooth. Let's exhibit a particularly explicit value C . Consider the main case $n = 3$. Denote by A the well known operator $A = -P\Delta$, where P is the classical orthogonal projection of L^2 onto H , the closure in L^2 of smooth, compact supported functions in Ω . Let c_0 be such that

$$\|\nabla v\|_6 \leq c_0 \|Av\|, \quad \forall v \in D(A).$$

Note that, roughly speaking, the constant c_0 is like the constant c_1 in the Sobolev's embedding $\|v\|_6 \leq c_1 \|\nabla v\|$. We have shown that the constant

$$C = \frac{\mu}{2c_0}, \quad (24)$$

in equation (23), implies smoothness.

More recently, in the famous article [8], the authors succeed in proving that, in the whole space case, the assumption (22) alone guarantees smoothness of solutions. This was, for many years, one of the most challenging, and

difficult, open problems in the mathematical theory of Navier-Stokes equations. Later on, extensions to the boundary has been obtained. See [13] for the half-space case, and [10] for curved smooth boundaries. Proofs of the above results are quite involved. So, despite these strong improvements, a look at reference [5] could be of some interest, since the straightforward proof is particularly elementary.

Concerning the subject of the present notes, namely statements involving only $n - 1$ components of the velocity, we recall that the results proved in [5] were extended in reference [6] to the case in which the two conditions, (22) and (23), are merely required to \bar{u} . It would be not difficult to extend to the half-space case, under slip boundary conditions, the result stated in [6], by putting together the proof developed in this last reference and that shown in the present notes.

It would be interesting to study the following problem, even in the whole space case.

Open problem 4.1. To extend the main result proved in reference [8] to the "two components case".

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